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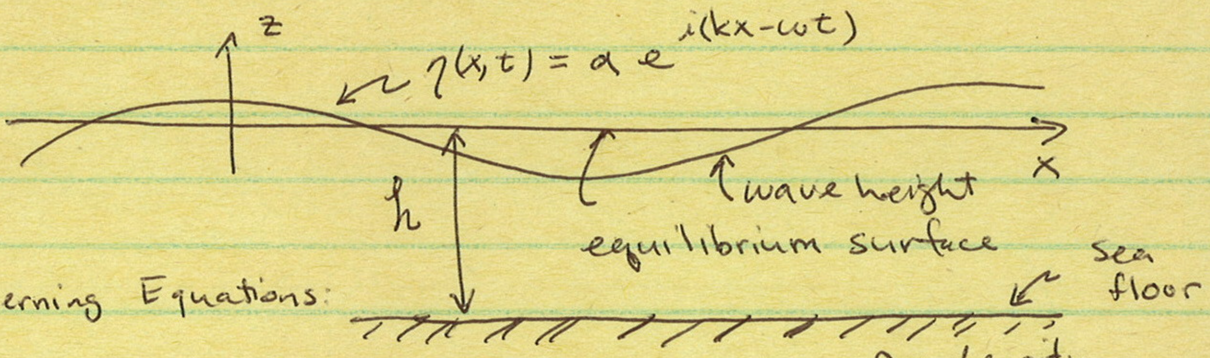
Tsunami Waves



Governing Equations:

ρ = density
$\vec{v}$ = velocity

Continuity: $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} = 0$$

Assuming incompressible medium

(1) $$\boxed{\vec{\nabla} \cdot \vec{v} = 0}$$

Momentum (Navier-Stokes) equation:

$$\rho \frac{D\vec{v}}{Dt} = \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} p + \rho \vec{g} + \mu \nabla^2 \vec{v}$$

Assume:
- viscous forces can be neglected $\mu \to 0$
- $\frac{\partial v}{\partial t} \gg v \cdot \frac{\partial v}{\partial x}$; second order terms neglected

(2) $\therefore$ $$\boxed{\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p + \rho \vec{g}}$$

Boundary Conditions

1) At bottom vertical velocity vanishes

$$v_z \Big|_{z=-h} = 0$$

2) pressure at sea surface is zero:

$$p\big|_{z=\eta} = 0$$

$$p\big|_{z=\eta} = p_{z=0} + \eta \frac{\partial p}{\partial z}\Big|_{z=0} + + +$$

$$p\big|_{z=\eta} \cong p_{z=0} = 0.$$

3) Velocity matches wave motion at free surface

$$\frac{D\eta}{Dt} = v_z\Big|_{z=\eta}$$

$$\frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} = v_z\Big|_{z=0} + \eta \frac{\partial v_z}{\partial z} + + +$$

Retaining only first order terms:

$$\frac{\partial \eta}{\partial t} \cong v_z\Big|_{z=0}$$

Summary:

(1) $\vec{\nabla} \cdot \vec{v} = 0$

(2) $\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p + \rho \vec{g}$

(3a) $v_z\big|_{z=-h} = 0$

(3b) $p\big|_{z=0} = 0$

(3c) $v_z\big|_{z=0} = \frac{\partial \eta}{\partial t}$

Method of Solution:

Define a velocity potential, ϕ, s.t.

$$\vec{V} = \vec{\nabla}\phi \iff V_i = \frac{\partial \varphi}{\partial x_i}$$

(4) $$\therefore \vec{\nabla}\cdot\vec{V} = \vec{\nabla}\cdot\vec{\nabla}\varphi = \boxed{\nabla^2\phi = 0}$$

Also $$\rho\, \vec{\nabla}\frac{\partial \varphi}{\partial t} = -\vec{\nabla} p - \rho g \hat{e}_z$$

Integrate in the z- direction:

(5) $$\boxed{\rho \frac{\partial \varphi}{\partial t} = -p - \rho g z}$$

B.C.'s (6a) $$\boxed{\left.\frac{\partial \varphi}{\partial z}\right|_{z=-h} = 0}$$ on sea bottom.

(6b) $$\boxed{\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} \quad z=0}$$ on sea surface

$p|_{z=\eta} = 0$ insert into

$$\cancel{\rho} \frac{\partial \varphi}{\partial t} = -\cancel{\rho} g \eta \quad \text{on } z=0$$

(6c) $$\boxed{\frac{\partial \varphi}{\partial t} = -g\eta \quad \text{at } z=0}$$

Wave height

$$\eta(x,t) = a\, e^{i(kx-\omega t)}$$

Choose $\quad \varphi(x,z,t) = f(z)\, e^{i(kx-\omega t+\Theta_0)}$

initial phase

Insert into $\quad \nabla^2\varphi = 0 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$

$$\left[-k^2 f(z) + \frac{d^2 f}{dz^2}\right] e^{i(kx-\omega t+\Theta_0)} = 0$$

(7) $\quad \therefore \quad \frac{d^2 f}{dz^2} - k^2 f = 0.$

Solution: $\quad f(z) = A\, e^{kz} + B\, e^{-kz}$

Now inforce boundary conditions (6a) & (6b)

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=-h} = k\left(A\, e^{-kh} - B\, e^{+kh}\right) e^{(ikx-\omega t+\Theta_0)} = 0.$$

$$\Rightarrow \quad A\, e^{-kh} - B\, e^{kh} = 0$$

(8) $\quad B = A\, e^{-2kh}$

Second boundary condition,

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t}$$

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = k\ (A-B)\ e^{i(kx-\omega t+\theta_0)}$$

$$\left.\frac{\partial \eta}{\partial t}\right|_{z=0} = -i\omega a\ e^{i(kx-\omega t)}$$

$$\therefore \quad k(A-B)\ e^{i\theta_0} = -i\omega a$$

$$\theta_0 = -\frac{\pi}{2} \qquad e^{i\theta_0} = i\sin(-\tfrac{\pi}{2}) = -i$$

$$\therefore \qquad A-B = \frac{\omega a}{k}$$

Combine with (8)

$$A\left(1-e^{-2kh}\right) = \frac{\omega a}{k}$$

$$(9) \qquad A = \frac{\omega a}{k(1-e^{-2kh})} \qquad\qquad B = \frac{a\omega e^{-2kh}}{k\left(1-e^{-2kh}\right)}$$

Note for $kh \to \infty$ $\quad A \to \omega a/k \qquad B \to 0$

Thus:

$$\varphi(x,z,t) = \frac{a\omega}{k(1-e^{-2kh})}\left[e^{kz} + e^{-2kh}e^{-kz}\right] e^{i\left(kx-\omega t-\frac{\pi}{2}\right)}$$

multiply top & bottom by e^{kh}

$$\varphi(x,z,t) = \frac{a\omega}{k} \frac{\left[e^{k(h+z)} + e^{-k(h+z)}\right]}{\left[e^{kh} - e^{-kh}\right]} e^{i(kx-\omega t-\frac{\pi}{2})}$$

OR:

$$\boxed{\varphi(x,z,t) = \frac{a\omega}{k} \frac{\cosh[k(h+z)]}{\sinh(kh)} e^{i(kx-\omega t-\frac{\pi}{2})}}$$

Note, we have now basically solved for the velocity field using only 2 of the 3 boundary conditions!
since $\vec{v} = \vec{\nabla}\varphi$

Final B.C. is $\left.\frac{\partial\varphi}{\partial t}\right|_{z=0} = -g\eta$

$$\therefore -i\frac{a\omega^2}{k} \frac{\cosh(kh)}{\sinh(kh)} e^{i(kx-\omega t)} \underbrace{e^{-i\frac{\pi}{2}}}_{-i} = -ga\, e^{i(kx-\omega t)}$$

$$\therefore \frac{\omega^2}{kg} \frac{\cosh(kh)}{\sinh(kh)} = 1$$

$$\omega^2 = kg\tanh(kh)$$

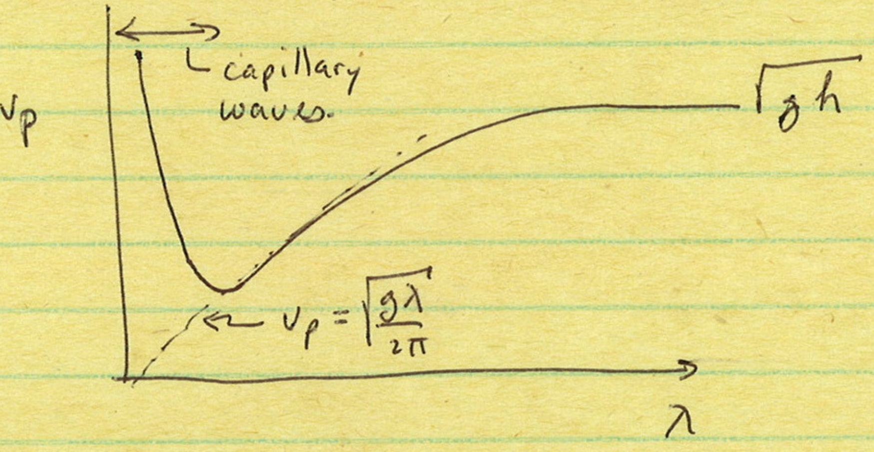
$$\boxed{\begin{gathered}\omega = \sqrt{kg\tanh(kh)} \\ V_\varphi = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)} = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)}\end{gathered}}$$

$$v_p = \sqrt{\frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right)}$$

for $\frac{2\pi h}{\lambda} \gg 1$ (deep water) short λ $\qquad v_p = \sqrt{\frac{g\lambda}{2\pi}}$

for $\frac{2\pi h}{\lambda} \ll 1$ (shallow water) long λ $\qquad v_p = \sqrt{\frac{g\lambda}{2\pi} \times \frac{2\pi h}{\lambda}} = \sqrt{gh}$

For tsunami waves $\boxed{v_p \cong \sqrt{gh}}$



Eigenfunctions:

$$\vec{v} = \vec{\nabla} \phi$$

$$\varphi = \frac{a\omega}{k} \frac{\cosh k(h+z)}{\sinh (kh)} e^{i(kx - \omega t - \frac{\pi}{2})}$$

$$V_z = \frac{a\omega \cancel{k}}{\cancel{k}} \frac{\sinh k(h+z)}{\sinh(kh)} e^{i(kx-\omega t-\frac{\pi}{2})}$$

$$V_x = \frac{a\omega}{\cancel{k}} \frac{\cosh k(h+z)}{\sinh(kh)} \times \cancel{ik}^{\,0} e^{i(kx-\omega t)} \cancel{e^{-i\frac{\pi}{2}}}^{\,0}$$

$$V_x = a\omega \frac{\cosh k(h+z)}{\sinh(kh)} e^{i(kx-\omega t)}$$

To determine displacements, integrate w.r.t. time.

$$V_z = \frac{\partial u_z}{\partial t} = a\omega \frac{\sinh k(h+z)}{\sinh(kh)} e^{i(kx-\omega t-\frac{\pi}{2})}$$

$$u_z = a\cancel{\omega} \frac{\sinh k(h+z)}{\sinh(kh)} \times \frac{\cancel{i}}{\cancel{\omega}} e^{i(kx-\omega t)} \cancel{e^{-i\frac{\pi}{2}}}$$

$$\boxed{u_z = a \frac{\sinh k(h+z)}{\sinh(kh)} e^{i(kx-\omega t)}}$$

$$V_x = \frac{\partial u_x}{\partial t} = a\cancel{\omega} \frac{\cosh k(h+z)}{\sinh(kh)} \frac{i}{\cancel{\omega}} e^{i(kx-\omega t)}$$

$$\boxed{u_x = a \frac{\cosh k(h+z)}{\sinh(kh)} e^{i(kx-\omega t+\frac{\pi}{2})}}$$

To write in the form of Ward

note

$$\omega^2 \cosh(kh) = gk \sinh(kh)$$

from dispersion relation. Thus

$$\boxed{\begin{aligned} u_z &= \frac{agk}{\omega^2}\left[\frac{\sinh k(h+z)}{\cosh(kh)}\right] e^{i(xk-\omega t)} \\ u_x &= \frac{agk}{\omega^2}\left[\frac{\cosh k(h+z)}{\cosh(kh)}\right] e^{i(kx-\omega t+\frac{\pi}{2})} \end{aligned}}$$

At small wavelengths ($kh \to \infty$), both [ ] terms go to e^{kz}. Circular orbits that decay exponentially with depth!

ie, $u_z \sim \dfrac{e^{k(h+z)} - e^{-k(h+z)}}{e^{kh} + e^{-kh}}$

limit as $kh \to \infty \Rightarrow \dfrac{e^{k(h+z)} - e^{-k(h+z)}}{e^{kh}} = e^{kz} - e^{-kh} e^{-(z+h)}$

in the limit $u_z \rightsquigarrow e^{kz}$ since $0 < e^{-(z+h)k} < 1$

For <u>long wavelengths</u>

$$kh \to 0$$

$$\cosh(kh) \to 1$$

$\therefore$ $u_z \sim \sinh k(h+z)$ $\qquad$ $u_x \sim \cosh k(h+z)$

$$u_x \sim e^{kh} e^{kz} + e^{-kh} e^{-kz} \qquad \lim_{kh \to 0} \to e^{kz} + e^{-kz} \to \cosh(kz)$$

note for $-h \; z < 0$ $\quad \cosh(kz) \to \cosh(0) = 1$

$u_z \sim \lim_{kh \to 0} \to \sinh(kz)$. For $-h < z < 0$ $\quad \sinh(kz) \to 0.$

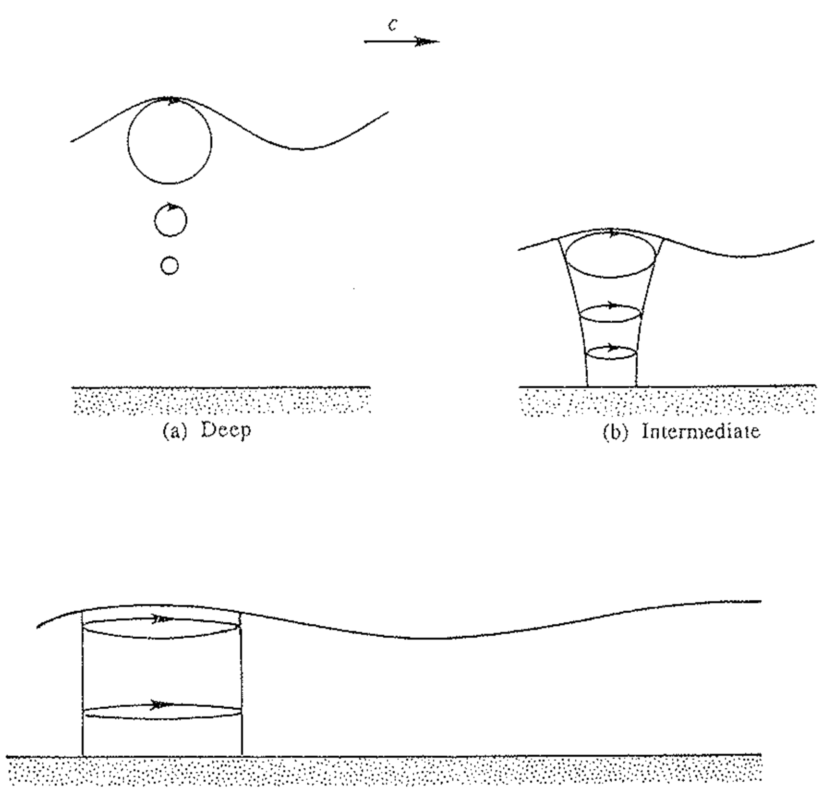


Fig. 7.6 Particle orbits of wave motion in deep, intermediate, and shallow seas.

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Tsunami Source

$$\nabla^2 \phi = 0$$

B.C's.
$$\left.\begin{aligned} \frac{\partial \varphi}{\partial z} &= \frac{\partial \eta}{\partial t} \\ \frac{\partial \varphi}{\partial t} &= -g\eta \end{aligned}\right\} \text{ on } z=0$$

$$V_z = \frac{\partial \varphi}{\partial z} = \frac{\partial F(x,t)}{\partial t} \Big\} \text{ on } z=-h$$ sea bottom displacement.

Take Laplace & Fourier Transforms:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, ds \; ; \quad \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

(1) $\nabla^2 \bar{\hat{\phi}} = \boxed{\dfrac{d^2 \bar{\hat{\varphi}}}{dz^2} - k^2 \bar{\hat{\varphi}} = 0}$

(2) B.C's $\dfrac{d\bar{\hat{\varphi}}}{dz} = s\bar{\hat{\eta}}$ } on $z=0$

(3) $s\bar{\hat{\varphi}} = -g\bar{\hat{\eta}}$

(4) $\dfrac{d\hat{\varphi}}{dz} = \bar{\hat{F}}$. on $z=-h$ assuming $F(x,t) = F(x)\,H(t)$ Heavyside step function

From (2) & (4) eliminate $\bar{\hat{\eta}}$

$$\eta = \frac{1}{s}\frac{d\bar{\hat{\varphi}}}{dz} = -\frac{s\bar{\hat{\varphi}}}{g} \Rightarrow \boxed{\frac{d\bar{\hat{\varphi}}}{dz} + \frac{s^2}{g}\hat{\varphi} = 0 \quad \text{on } z=0}$$

General soln for $\hat{\varphi}$

$$\hat{\varphi}(k) = A \cosh k(z+h) + B \sinh k(z+h)$$

$$\frac{d\hat{\varphi}}{dz} = Ak \sinh k(z+h) + Bk \cosh k(z+h)$$

$$\frac{d^2\hat{\varphi}}{dz^2} = Ak^2 \cosh k(z+h) + Bk^2 \sinh k(z+h)$$

BC #1: on $z=0$

$$gk\left[A \sinh k(\cancel{z}+h) + B \cosh k(\cancel{z}+h)\right] + s^2\left[A \cosh k(\cancel{z}+h) + B \sinh k(\cancel{z}+h)\right] = 0$$

$$0 = A\left[gk \sinh(kh) + s^2 \cosh(kh)\right] + B\left[gk \cosh(kh) + s^2 \sinh(kh)\right]$$

$$\therefore A = -B \frac{gk \cosh(kh) + s^2 \sinh(kh)}{gk \sinh(kh) + s^2 \cosh(kh)}$$

BC #2 $\quad \left.\frac{d\varphi}{dz}\right|_{z=-h} = \hat{\bar{F}} = Bk \Rightarrow B = \hat{\bar{F}}/k$

$$\therefore A = -\frac{H}{k} \frac{gk + s^2 \tanh(kh)}{s^2 + gk \tanh(kh)}$$

$$\therefore \quad \bar{\hat{\varphi}} = \frac{\bar{\hat{F}}}{k} \left\{ \frac{(s^2 + gk\tanh(kh))\sinh k(z+h) - (gk + s^2\tanh(kh)) \times \cosh k(z+h)}{s^2 + gk\tanh(kh)} \right\}$$

Note the numerator is:

$$gk\left[\sinh k(z+h)\frac{\sinh(kh)}{\cosh(kh)} - \cosh k(z+h)\right] +$$

$$s^2\left[\sinh k(z+h) - \cosh k(z+h)\frac{\sinh(kh)}{\cosh(kh)}\right]$$

$$= \frac{1}{\cosh(kh)}\left\{ gk\left[\sinh k(z+h)\sinh(kh) - \cosh k(z+h)\cosh(kh)\right] + s^2\left[\sinh k(z+h)\cosh(kh) - \cosh k(z+h)\sinh(kh)\right]\right.$$

$$= \frac{1}{\cosh(kh)}\left[-gk\cosh(kz) + s^2\sinh(kz)\right]$$

$$\therefore \quad \boxed{\bar{\hat{\varphi}}(k,s) = \frac{\bar{\hat{F}}(k)}{k\cosh(kh)}\left[\frac{s^2\sinh(kz) - gk\cosh(kz)}{s^2 + gk\tanh(kh)}\right]}$$

Free surface displacement from B.C. (2)

$$\bar{\hat{\eta}} = \frac{1}{s}\frac{d\bar{\hat{\varphi}}}{dz}\bigg|_{z=0}$$

$$\therefore \quad \bar{\hat{\eta}} = \frac{1}{\cancel{s}} \times \frac{\bar{\hat{F}}(k)}{\cancel{k}\cosh(kh)} \; \frac{s^{\cancel{2}}\cancel{k}}{s^2 + gk\tanh(kh)}$$

but $\quad gk\tanh(kh) = \omega^2 \quad$ From dispersion relation.

$$\therefore \quad \boxed{\bar{\hat{\eta}} = \frac{\bar{\hat{F}}(k)}{\cosh(kh)} \; \frac{s}{s^2 + \omega^2}}$$

Now invert the transforms. First the Laplace transform

$$\hat{\eta}(k,t) = \frac{\hat{F}(k)}{\cosh(kh)} \times \underbrace{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{s\,e^{st}\,ds}{(s+i\omega)(s-i\omega)}}_{\frac{\cancel{i\omega}\,e^{i\omega t}}{2\cancel{i\omega}} + \frac{\cancel{+i\omega}\,e^{-i\omega t}}{+2\cancel{i\omega}} = \cos(\omega t)}$$

$$\therefore \quad \hat{\eta}(k,t) = \frac{\hat{F}(k)}{\cosh(kh)} \times \frac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

Now invert the Fourier transform

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{F}(k)}{\cosh(kh)}\, e^{ikx} \times \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) dk$$

$$\boxed{\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\hat{F}(k)\,dk}{\cosh(kh)} \left[e^{i(kx-\omega t)} + e^{+i(kx+\omega t)}\right]}$$