Transient Pressure Analysis in Strip Reservoirs with Linear Skin Discontinuities

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Discontinuities in reservoirs are important considerations when analysing pressure drawdown and buildup tests. The changes exhibited in test results can reveal important geometrical features of the reservoir boundary. And, based on the pressure transient analysis, conclusions as to the preferred treatment of the reservoir can be drawn.

A mathematical model of analysing pressure transient tests for linear skin discontinuities is presented. The case of a strip reservoir which is unbounded horizontally in one direction and bounded with impermeable barriers vertically and containing a linear skin discontinuity is proposed. The problem is solved using the two dimensional diffusion equation with successive integral transformations.

Confirmation of the solution is demonstrated by the early time line source pressure response and the late time linear flow pressure response. Application to well testing can be made with superposition of the constant rate fluid production and with matching times to events such as fluid barriers with the solutions.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>v</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. LITERATURE SURVEY</td>
<td>3</td>
</tr>
<tr>
<td>III. STATEMENT OF PROBLEM</td>
<td>6</td>
</tr>
<tr>
<td>IV. ANALYTICAL, SOLUTION</td>
<td>7</td>
</tr>
<tr>
<td>V. DISCUSSION</td>
<td>21</td>
</tr>
<tr>
<td>VI. COMPUTATIONAL PROCEDURES</td>
<td>30</td>
</tr>
<tr>
<td>VII. CONCLUSIONS AND RECOMMENDATIONS</td>
<td>31</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>32</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>33</td>
</tr>
<tr>
<td>APPENDIX A. DELIMITING SOLUTION FOR LINE SOURCE CONCURRENCE</td>
<td>35</td>
</tr>
<tr>
<td>APPENDIX B. COMPUTER PROGRAMS</td>
<td>40</td>
</tr>
<tr>
<td>APPENDIX C. SELECTED DATA</td>
<td>53</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 1 : Schematic diagram for the linear skin discontinuity system .............................................. 7

Figure 2 : Semilog plot depicting the behavior of the pressure function during numerical inversion........................................................................................................................................................................... 22

Figure 3 : Logarithmic plot of $p_D$ vs. $t_D$ for values generated at $y^*_D = 20$ in the Fourfirst and Stehfist Programs........................................................................................................... 23

Figure 4 : Logarithmic plot of the sum of line sources with a well located in the center of the system. $p_D = \frac{1}{2} \sum_{i=1}^{n} E_i \left( \frac{y^*_D b_i^2}{4t_D} \right) - \frac{1}{2} E_i \left( \frac{1}{4t_D} \right)$ where $y^*_D$ ranges from 10 to 10000........................................................................................................................................................................... 25

Figure 5 : Logarithmic plot of $p_D$ vs. $t_D$ for composite reservoir with $y^*_D$ varying from 10 to 200........................................................................................................................................................................... 26

Figure 6 : Logarithmic plot of $p_D$ vs. $t_D$ shifted data from the composite system. ........................................................................................................................................................................... 27

Figure 7 : Logarithmic plot for case of infinite and zero boundary skin........................................... 28

Figure 8 : Logarithmic plot for case of infinite and unit mobility....................................................... 29

Figure 9 : Flowchart for Fourfirst Program ........................................................................................ 41

Figure 10 : Flowchart for Stehfist Program .................................................................................... 47
LIST OF TABLES

Table C-1 : Selected Data from Stehfirst and Fourfirst Program Evaluation..............54
Table C-2 : Selected Data from Composite Reservoir -- Nonshifted ........................55
Table C-3 : Selected Data from Composite Reservoir -- Shifted.............................56
Table C-4 : Selected Data for Varying Mobility Ratios and Skin..........................57
Table C-5 : The Line Source Solution.....................................................................58
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>v</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. LITERATURE SURVEY</td>
<td>3</td>
</tr>
<tr>
<td>III. STATEMENT OF PROBLEM</td>
<td>6</td>
</tr>
<tr>
<td>IV. ANALYTICAL SOLUTION</td>
<td>7</td>
</tr>
<tr>
<td>V. DISCUSSION</td>
<td>21</td>
</tr>
<tr>
<td>VI. COMPUTATIONAL PROCEDURES</td>
<td>30</td>
</tr>
<tr>
<td>VII. CONCLUSIONS AND RECOMMENDATIONS</td>
<td>31</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>32</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>33</td>
</tr>
<tr>
<td>APPENDIX A. DELIMITING SOLUTION FOR LINE SOURCE CONCURRENCE</td>
<td>35</td>
</tr>
<tr>
<td>APPENDIX B. COMPUTER PROGRAMS</td>
<td>40</td>
</tr>
<tr>
<td>APPENDIX C. SELECTED DATA</td>
<td>53</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

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Figure 2: Semilog plot depicting the behavior of the pressure function during numerical inversion. ................................................................. 22

Figure 3: Logarithmic plot of \( p_D \) vs. \( t_D \) for values generated at \( y^D = 20 \) in the Fourfirst and Stehfirst Programs. ................................................................. 23

Figure 4: Logarithmic plot of the sum of line sources with a well located in the center of the system. \( p_D = \frac{1}{2} \sum_{\mu=1}^{\infty} Ei(-\frac{y^\mu t}{4t_D}) - \frac{1}{2} Ei(-\frac{1}{4t_D}) \)
where \( y^D \) ranges from 10 to 10000. ................................................................. 25

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Figure 6: Logarithmic plot of \( p_D \) vs. \( t_D \) shifted data from the composite system. ........ 27

Figure 7: Logarithmic plot for case of infinite and zero boundary skin. ..................... 28

Figure 8: Logarithmic plot for case of infinite and unit mobility. ............................... 29

Figure 9: Flowchart for Fourfirst Program ................................................................. 41

Figure 10: Flowchart for Stehfirst Program ................................................................. 47
LIST OF TABLES

Table C-1 : Selected Data from Stehfirst and Fourfirst Program Evaluation ........... 54
Table C-2 : Selected Data from Composite Reservoir Nonshifted ................................ 55
Table C-3 : Selected Data from Composite Reservoir Shifted ...................................... 56
Table C-4 : Selected Data for Varying Mobility Ratios and Skin .................................. 57
Table C-5 : The Line Source Solution ........................................................................ 58
I. INTRODUCTION

Pressure transient theory has been classically based on reservoir homogeneity. However, practice has shown that it is quite important to consider heterogeneities and their effect on transient pressure behavior. Heterogeneities such as natural fractures, faults, gas caps, and man-made anomalies such as hydraulic fractures or injection well tests exemplify alterations in transient pressure behavior. In order to understand such anomalies, pressure buildup and drawdown tests have become valuable resources for explaining drainage limits, formation and wellbore storage, average permeability, and the presence of flow discontinuities.

In recent years, flow barriers have attracted industry and academic research interest. Areas such as the North Slope of Alaska which exhibits a highly faulted structure, in addition to a major gas cap, and the tight sands of Colorado which contain hydraulic fractures typify complex discontinuities in reservoir structure. Notwithstanding educational interests are economical interests at stake in producing these fields. As a result, more information is being demanded from well tests such that reservoir development can be planned from strategy rather than conjecture. Alternately, research has the challenge to establish theoretical bases in pressure transient work for defining fluid behavior with internal flow barriers.

Studies concerning flow discontinuities have centered on internal circular and linear boundaries with either constant pressure and no flow or constant rate systems. Linear boundaries have been analysed quite extensively with assumptions of a constant change in pressure and constant mobility across the boundary. In work to date, the concept of a system with a penetrable flow boundary compounded with a skin has not been analysed. This type of system would apply to reservoirs exhibiting the heterogeneities described earlier. For example, the gas cap showing high compressibility compared to the otherwise uniform formation compressibility may produce a discontinuity with skin.

The research presented in this paper involves a semi-infinite reservoir with a line source well located some distance from a linear skin discontinuity. Constant rate fluid production is
assumed. The pressure drop associated with the skin will be a function of the flow rate and change in pressure as established from behavior of the line source near the discontinuity. Measurement of the skin with varying distances along the linear boundary from the line source provides data to construct type curves.
II. LITERATURE SURVEY

Since the late 1940s attention has been given to the use of transient pressure measurements in wells as a device to investigate the behavior of aquifers and petroleum reservoirs. Theoretically, pressure buildup and drawdown results describe the nature of the system from whence they are drawn. Results, based on the soundness of this theory, are then made the basis for developing reservoir description. The assessment of buildup and drawdown tests have focused on two general conditions: constant pressure boundaries and/or no-flow boundaries. The research presented in this paper considers a semi-infinite, constant rate reservoir containing a linear skin discontinuity; the skin being a result of changes in fluid properties caused by fluid-fluid interfaces or structural discontinuities.

In a classic text by Carslaw and Jaeger [1959], the solution for a constant flow rate line source well inside an infinite reservoir was presented. Van Everdingen and Hurst [1949] applied the Laplace transformation to this condition and showed the solution for a finite radius well in an infinite system. The line source referred to in these papers is defined by Ramey, Kumar, and Gulati [1973] as a well having a vanishing radius in an infinitely large system. The formation is considered at constant permeability, porosity, and thickness while the fluid is considered producing at a constant rate with constant compressibility and viscosity. Also, pressure gradients are small such that the square of the gradient may be neglected in the flow system. Matthews and Russell [1967], among others, define the infinite reservoir as an infinite acting reservoir where there is negligible pressure depletion at the outer boundary caused by a well located some distance from that boundary.

Completing the reservoir definition calls for understanding the internal boundaries, if they exist. As described by Muskat [1937], inner discontinuities are of practical interest when abrupt changes in permeability are noticed with respect to geometrical boundary crossover. Muskat also showed two conditions that must be satisfied when considering discontinuities: equal pressures continuing across the surface of discontinuity and, based on Darcy’s Law, the normal
velocity also continuous across the discontinuity. As a result, the pressure regime will correspond to a composite flow system where the two varying permeability regions are connected physically and geometrically. Bixel, Larkin, and van Poollen [1963] presented a general analysis of the linear discontinuity in an infinite system. Mathematically, they described two regions with independently homogeneous fluid and rock properties. Bixel, et al. extended Muskat's assumptions and developed a method for measuring pressure changes at producing wells some distance, which they also calculated, from the discontinuity. Others, such as Standing [1964], Gibson and Campbell [1970], Prasad [1975], Fenske [1984], and Yaxley [1985] developed supporting results for calculating distance to a discontinuity.

The concept of measuring the pressure effects on a well from an inner boundary have been studied by numerous authors. Elkins and Skov [1960] used the approach of an anisotropic media containing natural fractures and correlated results from pressure transient curves to injection tests with good agreement. Similarly, Cinco, Samaniego, and Dominguez [1976] studied the natural fracture effect on transient pressure but treated the fracture as an internal finite linear boundary. Davis and Hawkins [1963] looked at another form of internal boundary when they studied effects on pressure transients near a fault, (no-flow boundary). Hurst [1960] considered the interference between oil fields similar to that of a well being affected by some other fluid producer. In Hurst's discussion, it is shown that the pressure drop occurring at a distance, r, is dependent upon the physical parameters existing in the formation at that point. In essence, interference seen in the subject well is accounted for via two parameters, distance moved and rate of voidage, q̇, with other formation parameters such as thickness and permeability held constant. As developed earlier, Hurst [1960] simplified the complexities formerly associated with identifying and determining reservoir pressure in the material balance relationship by treatment with the Laplace transformation. The inversion of which showed pressure as an explicit function of those factors contributing to its change.

Horner [1951]; Matthews, Brons, and Hazebroek [1954]; Collins [1961]; and, Earlbougher, Ramey, Miller, and Mueller [1968] used the method of images for generating the
effects of linear boundaries. Stullman [1952] and Tiub and Kumar [1980] generated log-log type curves based on behavior from imaged source-sink wells generating the effects of both constant pressure and no-flow linear boundaries. Internal circular discontinuities have been studied by Sageev [1983]. Results from this study show a well produced near a no-flow boundary developing two different pressure regions. One region, containing the producer displaying a pressure drop higher than the line source and the other region showing a pressure drop lower than the line source.

The method of images cannot be applied to this problem since the linear skin discontinuity contains neither a constant pressure nor a no-flow boundary. In this research, a permeable skin discontinuity is studied. As such, the approach is similar to the mathematical method developed by Bizel, et al. [1963] with variations to their inner boundary condition to include the effect of skin. Also, the method of Goode and Thumbymayagam [1985] is used where a set of equations describing transient pressure in horizontal wells were derived using successive integral transformations.
III. STATEMENT OF PROBLEM

The objective of this research involves mathematically defining transient pressure in a strip reservoir with a linear skin discontinuity. The characteristics of the skin are measured in relation to a line source well located some fixed distance from the discontinuity. Results from the research provide a basis in transient testing for detecting a semi-permeable discontinuity caused by rock changes or fluid interfaces.
IV. ANALYTICAL SOLUTION

The problem is posed in two dimensions as a strip with a linear skin discontinuity approximated by a vertical plane. An infinite conductivity well is located in region I which produces a fluid of constant compressibility. The initial reservoir pressure is constant. Transport of fluids vertically is prohibited and unlimited horizontally. (see figure 1).

Figure 1: Schematic diagram for the linear skin discontinuity system

The diffusivity, $\eta$, and mobility, $\lambda$, are defined for each region as:

$$
\eta_1 = \left[ \frac{k}{\rho \mu c} \right]_1 \quad \lambda_1 = \left[ \frac{k}{\mu} \right]_1 \\
\eta_2 = \left[ \frac{k}{\rho \mu c} \right]_2 \quad \lambda_2 = \left[ \frac{k}{\mu} \right]_2
$$

The partial differential equations governing the isothermal fluid flow are as follows:

For the first region, $x>0$, containing the active constant rate well,
For the second region, \(x<0\),
\[
\frac{\partial p_2}{\partial t} = \eta_2 \left[ \frac{\partial^2 p_2}{\partial x^2} + \frac{\partial^2 p_2}{\partial y^2} \right] = 0
\]  

The initial, boundary, and interface conditions for the problem are:

**Initial Conditions:**
\[
p_1(x,y,0) = P_i \\
p_2(x,y,0) = P_i
\]

**Boundary Conditions:**
\[
p_1(\infty, y, t) = p_2(-\infty, y, t) = P_i
\]

For \(x>0, x<0\):
\[
\left. \frac{\partial p_1}{\partial y} \right|_{y=0} = \left. \frac{\partial p_2}{\partial y} \right|_{y=0} = 0
\]

**Interface Conditions:**
\[
\begin{bmatrix} \frac{\partial p}{\lambda \partial x} \end{bmatrix}_1 = \begin{bmatrix} \frac{\partial p}{\lambda \partial x} \end{bmatrix}_2
\]

and for the skin discontinuity,
\[
\frac{\partial p_2(0,y,t)}{\partial x} = \frac{-1}{s_1} \left[ -p_2(0,y,t) + p_1(0,y,t) \right]
\]

with, \(s=1\), a characteristic length.
To facilitate solution of the problem, dimensionless variables are defined and substituted into the partial differential equations and boundary conditions.

Dimensionless Variables:

\[ p_{D_1} = \frac{k_1 h}{\mu_1 q} \left[ p_1 - p(x,y,t) \right] \]  \hspace{1cm} (9)

\[ p_{D_2} = \frac{k_2 h}{\mu_2 q} \left[ p_1 - p(x,y,t) \right] \]  \hspace{1cm} (10)

\[ x_D = \frac{x}{r_w} \]  \hspace{1cm} (11)

\[ y_D = \frac{y}{r_w} \]  \hspace{1cm} (12)

\[ t_D = t_{D_1} = \frac{k}{\phi \mu c_t} \left\{ \frac{t}{r_w^2} \right\} \]  \hspace{1cm} (13)

Dimensionless Equations:

For \( x_D > 0 \) :

\[ \frac{\partial p_{D_1}}{\partial t_D} = \frac{\partial^2 p_{D_1}}{\partial x_D^2} + \frac{\partial^2 p_{D_1}}{\partial y_D^2} - \delta(x_D-a)\delta(y_D-b) \]  \hspace{1cm} (14)

For \( x_D < 0 \) :

\[ \frac{\eta_1}{\eta_2} \frac{\partial p_{D_2}}{\partial t_D} = \frac{\partial^2 p_{D_2}}{\partial x_D^2} + \frac{\partial^2 p_{D_2}}{\partial y_D^2} \]  \hspace{1cm} (15)

where \( t_D \) is defined in terms of regions I permitting a diffusivity relation between region I and II.
Dimensionless Boundary Conditions:

Initial conditions:

\[ p_{D_1}(x_D, y_D, 0) = 0 \] \hspace{1cm} (16)
\[ p_{D_2}(x_D, y_D, 0) = 0 \] \hspace{1cm} (17)

Outer boundary conditions:

\[ p_{D_1}(\infty, y_D, t_D) = p_{D_2}(-\infty, y_D, t_D) = 0 \] \hspace{1cm} (18)

For \( x_D > 0, \ x_D < 0 \):

\[ \frac{\partial p_{D_1}}{\partial y_D} \bigg|_{y_D^o} = \frac{\partial p_{D_1}}{\partial y_D} \bigg|_0 = \frac{\partial p_{D_2}}{\partial y_D} \bigg|_{y_D^o} = \frac{\partial p_{D_2}}{\partial y_D} \bigg|_0 = 0 \] \hspace{1cm} (19)

Interface Conditions:

\[ \frac{\partial p_{D_1}}{\partial x_D} (0, y_D, t_D) = \frac{\lambda_2}{\lambda_1} \frac{\partial p_{D_2}}{\partial x_D} (0, y_D, t_D) \] \hspace{1cm} (20)

and,

\[ \frac{\partial p_{D_1}}{\partial x_D} (0, y_D, t_D) = \frac{1}{s} \left[ -p_{D_1}(0, y_D, t_D) + p_{D_1}(0, y_D, t_D) \right] \] \hspace{1cm} (21)

The next step in solving the problem is with a transformation in time and in the \( y \) coordinate,

the variable transform for \( y_D \) is defined as,

\[ Y_D = \frac{\pi y_D}{y_D^o} = v y_D \] \hspace{1cm} (22)

where

\[ v = \frac{A}{y_D^o} \] \hspace{1cm} (23)
From Churchill [1944], the Laplace transformation in time is defined as,

\[ L \left\{ f(t) \right\} = \int_0^\infty e^{-st} f(t) \, dt = \mathcal{F}(s) = \tilde{f}(s) \]  

(24)

and from Selby [1970], the Finite Fourier Cosine transformation in \( y \),

\[ F \left\{ G(y) \right\} = \int_0^\infty G(y) \cos(my) \, dy \]  

(25)

giving the dimensionless pressure to be evaluated as,

\[ \bar{p}_D = \int_0^{\gamma_D} \cos(m y_D) \int_0^{\infty} e^{-S_D p_D(x_D, y_D, t_D)} \, dt_D dy_D \]  

(26)

or, more simply as,

\[ \bar{p}_D = \int_0^{\gamma_D} \cos(m y_D) \bar{p}_D \, dy_D \]  

(27)

applying the Laplace transformation to equation 14 and 15 yields,

for \( x_D > 0 \):

\[ \frac{\partial^2 \bar{p}_{D_1}}{\partial x_D^2} + \nu^2 \frac{\partial^2 \bar{p}_{D_1}}{\partial y_D^2} - \frac{\delta(x_D-a) \delta(\frac{1}{\nu} Y_D-b)}{S} = S\bar{p}_{D_1} - p_{D_1(t=0)} \]  

(28)

applying the initial condition, equation 16, gives,

\[ \frac{\partial^2 \bar{p}_{D_1}}{\partial x_D^2} + \nu^2 \frac{\partial^2 \bar{p}_{D_1}}{\partial y_D^2} - \frac{\delta(x_D-a) \delta(\frac{1}{\nu} Y_D-b)}{S} = S\bar{p}_{D_1} \]  

(29)

for \( x_D < 0 \),

\[ \frac{\partial^2 \bar{p}_{D_2}}{\partial x_D^2} + \nu^2 \frac{\partial^2 \bar{p}_{D_2}}{\partial y_D^2} = \frac{\eta_1}{\eta_2} \bar{p}_{D_2} - p_{D_2(t=0)} \]  

(30)
applying the initial condition, equation 17, gives,
\[
\frac{\partial^2 \overline{p}_{D_2}}{\partial x_D^2} + \nu^2 \frac{\partial^2 \overline{p}_{D_2}}{\partial y_D^2} = \frac{\eta_1}{\eta_2} \overline{p}_{D_2}
\]  
(31)

Laplace transforming the boundary conditions,
\[
\overline{p}_{D_1}(\infty, y_D, S) = \overline{p}_{D_2}(-\infty, y_D, S) = 0
\]  
(32)

\[
\frac{\partial \overline{p}_{D_1}}{\partial y_D} \bigg|_{y_D^*} = \frac{\partial \overline{p}_{D_2}}{\partial y_D} \bigg|_{y_D^*} = \frac{\partial \overline{p}_{D_2}}{\partial y_D} \bigg|_{0} = 0
\]  
(33)

and, Laplace transforming the interface conditions,
\[
\frac{\partial \overline{p}_{D_1}}{\partial x_D} (0, y_D, S) = \frac{\lambda_2}{\lambda_1} \frac{\partial \overline{p}_{D_2}}{\partial x_D} (0, y_D, S)
\]  
(34)

\[
\frac{\partial \overline{p}_{D_1}}{\partial x_D} (0, y_D, S) = \frac{1}{s} \left[ -\overline{p}_{D_1}(0, y_D, S) + \overline{p}_{D_2}(0, y_D, S) \right]
\]  
(35)

Applying the Finite Fourier Cosine transformation and using the outer boundary condition, equation 33, to equations 29 and 31,
\[
\frac{d^2 \overline{p}_{D_1}}{dx_D^2} - \overline{p}_{D_1}(v^2 m^2 + S) = \frac{\delta(x_D-a)\cos(mb v)}{s}, \quad x_D > 0
\]  
(36)

\[
\frac{d^2 \overline{p}_{D_2}}{dx_D^2} - \overline{p}_{D_2}(v^2 m^2 + \frac{\eta_1}{\eta_2} S) = 0, \quad x_D < 0
\]  
(37)
The Finite Fourier Cosine transformation of the boundary conditions yield,
\[ \tilde{p}_{D_1}(\pm \infty, m, S) = 0 \] (38)

The Finite Fourier Cosine transformation of the interface conditions yields,
\[ \frac{d\tilde{p}_{D_1}}{dx_D}(0, m, S) = \frac{\lambda_2}{\lambda_1} \frac{d\tilde{p}_{D_2}}{dx_D}(0, m, S) \] (39)

\[ \frac{d\tilde{p}_{D_1}}{dx_D}(0, m, S) = \frac{-1}{s} \left[ -\tilde{p}_{D_2}(0, m, S) + \tilde{p}_{D_1}(0, m, S) \right] \] (40)

the General Solution to equation 37 is:
\[ \tilde{p}_{D_2} = Ce^{\sqrt{\frac{\eta_1}{\eta_2}} x_D} + De^{\sqrt{\frac{\eta_1}{\eta_2}} x_D} \] (41)

To find the constant, C, equation 38 will be used.
\[ \tilde{p}_{D_2}(\pm \infty, m, S) = 0 \]
yielding
\[ C = 0 \] (42)

Rewriting the general solution,
\[ \tilde{p}_{D_2} = De^{\sqrt{\frac{\eta_1}{\eta_2}} x_D} \] (43)

To solve equation 36, we take the Laplace transformation with respect to \( x_D \) where "u" is the Laplace variable.
defining the following substitutions:

\[ \alpha_1 = v^2 m^2 + S \]  

(44)

\[ \alpha_2 = v^2 m^2 + \frac{1}{112} S \]  

(45)

\[ \alpha_3 = \frac{\cos(\nu mb)}{S} \]  

(46)

Equation 36 becomes:

\[ u \left( u \overline{p}_{D_1} - \overline{p}_{D_1}(s,m,x_D=0) \right) - \frac{d\overline{p}_{D_1}}{ds_D}(s,m,x_D=0) - \alpha_1 \overline{p}_{D_1} = \alpha_3 e^{-\alpha u} \]  

(47)

and, rearranging, equation 47 gives,

\[ \frac{\alpha_3 e^{-\alpha u} + u \overline{p}_{D_1}(s,m,x_D=0) + \overline{p}_{D_1}(x_D=0)}{u^2 - \alpha_1} = \frac{\overline{p}_{D_1}}{u^2 - \alpha_1} \]  

(48)

simplifying,

\[ \frac{\alpha_3 e^{-\alpha u}}{u^2 - \alpha_1} + \frac{u \overline{p}_{D_1}(x_D=0)}{u^2 - \alpha_1} + \frac{\overline{p}_{D_1}(x_D=0)}{u^2 - \alpha_1} = \frac{\overline{p}_{D_1}}{u^2 - \alpha_1} \]  

(49)

From Churchill [1944], inversion of the following Laplace transforms are defined as,

\[ L^{-1} \left[ e^{-\alpha u} \right] = \delta(x_D-a) \]  

(50)

\[ L^{-1} \left[ \frac{u}{u^2 - \alpha_1} \right] = \cosh(\sqrt{\alpha_1} x_D) \]  

(51)
and,

\[ L^{-1} \left[ \frac{1}{u^2 - \alpha^2} \right] = \frac{1}{\sqrt{\alpha_1}} \sinh(\sqrt{\alpha_1} x_D) \]  

(52)

by the Convolution Integral from Kaplan [1981],

\[ L^{-1} \left[ \frac{\delta'' \alpha}{u^2 - \alpha^2} \right] = \frac{1}{\sqrt{\alpha_1}} \int_0^{x_D} \delta(z-a)\sinh(\sqrt{\alpha_1}(x_D-z)) \, dz \]  

(53)

Evaluating equation 53 for \( x_D < a \) gives,

\[ \frac{1}{\sqrt{\alpha_1}} \int_0^{x_D} \delta(z-a)\sinh(\sqrt{\alpha_1}(x_D-z)) \, dz = 0 \]  

(54)

and for \( x_D > a \),

\[ \frac{1}{\sqrt{\alpha_1}} \int_0^{x_D} \delta(z-a)\sinh(\sqrt{\alpha_1}(x_D-z)) \, dz = \frac{1}{\sqrt{\alpha_1}} \sinh(\sqrt{\alpha_1}(x_D-a)) \]  

(55)

Application of the inverse Laplace transformation to equation 49 yields:

For \( 0 < x_D < a \),

\[ \bar{p}_{D_1} = \bar{p}_{D_1}(x_D=0)\cosh(\sqrt{\alpha_1} x_D) + \bar{p}_{D_1}(x_D=0) \frac{1}{\sqrt{\alpha_1}} \sinh(\sqrt{\alpha_1} x_D) \]  

(56)

and if \( x_D > a \),

\[ \bar{p}_{D_1} = \frac{\alpha_3}{\sqrt{\alpha_1}} \sinh(\sqrt{\alpha_1}(x_D-a)) + \bar{p}_{D_1}(x_D=0)\cosh(\sqrt{\alpha_1} x_D) + \bar{p}_{D_1}(x_D=0) \frac{1}{\sqrt{\alpha_1}} \sinh(\sqrt{\alpha_1} x_D) \]  

(57)

Solving the problem defined in region 11, \( x_D < 0 \), is done by evaluation of the general solution obtained in equation 43.

Recall equation 43,
\[ \bar{P}_{D_2} = De^{\sqrt{\alpha_2} x_D} \quad ; \quad x_D < 0 \]

then differentiate with respect to \( x_D \) giving,

\[ \frac{d\bar{P}_{D_2}}{dx_D} = D \sqrt{\alpha_2} e^{\sqrt{\alpha_2} x_D} \quad (58) \]

Substituting equation 58 into 39 at \( x_D = 0 \) gives,

\[ \frac{d\bar{P}_{D_1}}{dx_D} (0, m, s) = \frac{\lambda_2}{\lambda_1} D \sqrt{\alpha_2} \quad (59) \]

and, similarly, substituting equation 58 into equation 40 at \( x_D = 0 \) gives,

\[ \frac{\lambda_2}{\lambda_1} D \sqrt{\alpha_2} = -\frac{1}{s} \left[ -D + \bar{P}_{D_1} (0, m, s) \right] \quad (60) \]

rearranging equation 60 yields,

\[ \bar{P}_{D_1} (0, m, s) = D \left[ 1 - s \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} \right] \quad (61) \]

Substituting equations 59 and 61 into equation 57 gives,

\[ \bar{P}_{D_1} = \frac{\alpha_3}{\sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} (x_D - a)} - e^{-\sqrt{\alpha_1} (x_D - a)} \right] \]

\[ + D \left[ 1 - s \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} \right] \left[ e^{\sqrt{\alpha_1} x_D} + e^{-\sqrt{\alpha_1} x_D} \right] \]

\[ + D \frac{\lambda_2}{\lambda_1} \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} x_D} - e^{-\sqrt{\alpha_1} x_D} \right] \quad (62) \]
Applying equation 38 and rearranging,

\[ D = \frac{-\frac{\alpha_3}{\sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} x_0} - \frac{e^{\sqrt{\alpha_1} x_0}}{2} \right]}{f + g} \]  

(63)

where \( f \) and \( g \) are defined as,

\[ f = \frac{1}{2} \left[ 1 - \frac{1}{\lambda_1} \sqrt{\alpha_2} \right] \left[ e^{\sqrt{\alpha_1} x_0} - \frac{e^{\sqrt{\alpha_1} x_0}}{2} \right] \]  

(64)

and

\[ g = \frac{\lambda_2}{\lambda_1} \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} x_0} - \frac{e^{\sqrt{\alpha_1} x_0}}{2} \right] \]  

(65)

rearranging equation 63 yields,

\[ D = \frac{-\alpha_3 e^{\sqrt{\alpha_1} (x_0 - s)}}{\sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} x_0} - \frac{e^{\sqrt{\alpha_1} x_0}}{2} \right] \left[ 1 - \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} + \frac{\lambda_2}{\lambda_1} \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} \right] \]  

(66)

simplifying equation 66 gives the definition of \( D \) in the general solution as,

\[ D = \frac{-\lambda_1 \alpha_3 e^{-\sqrt{\alpha_1}}}{\lambda_1 \sqrt{\alpha_1} - s\lambda_2 \sqrt{\alpha_2} \sqrt{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} \]  

(67)

Substituting equation 67 into equation 57 gives,

\[ \overline{D}_{\alpha_1} = \frac{\alpha_3}{\sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} (x_0 - s)} - \frac{e^{\sqrt{\alpha_1} (x_0 - s)}}{2} \right] \]

\[ - \frac{\lambda_1 \alpha_3 e^{-\sqrt{\alpha_1}}}{\lambda_1 \sqrt{\alpha_1} - s\lambda_2 \sqrt{\alpha_2} \sqrt{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} \]
- 18 -

\[
\begin{align*}
\frac{\lambda_2}{\lambda_1} \sqrt{a_2} \left[ e^{\sqrt{a_1} x_0} - e^{-\sqrt{a_1} x_0} \right] \\
+ \frac{\lambda_2}{\lambda_1} \sqrt{a_2} \left[ e^{\sqrt{a_1} x_0} - e^{-\sqrt{a_1} x_0} \right] \\
\end{align*}
\]

(68)

rearranging,

\[
\tilde{p}_{D_1} = \frac{\alpha_3}{2\sqrt{a_1}} \left[ e^{\sqrt{a_1} (x_0-a)} - e^{-\sqrt{a_1} (x_0-a)} \right] \\
- \frac{\lambda_1 \alpha_3}{2(\lambda_1 \sqrt{a_1} - s \lambda_2 \sqrt{a_2} \sqrt{a_1} + \lambda_2 \sqrt{a_2})} \\
\times \left[ e^{\sqrt{a_1} (x_0-a)} \left[ 1 - s \frac{\lambda_2}{\lambda_1} \sqrt{a_2} + \frac{\lambda_2}{\lambda_1} \sqrt{a_2} \right] \\
+ e^{-\sqrt{a_1} (x_0-a)} \left[ 1 - s \frac{\lambda_2}{\lambda_1} \sqrt{a_2} - \frac{\lambda_2}{\lambda_1} \sqrt{a_2} \right] \right] \\
\end{align*}
\]

(69)

simplifying,

\[
\tilde{p}_{D_1} = \frac{\alpha_3}{2\sqrt{a_1}} \left[ e^{\sqrt{a_1} (x_0-a)} - e^{-\sqrt{a_1} (x_0-a)} \right] - \frac{\alpha_3}{2\sqrt{a_1}} e^{\sqrt{a_1} (x_0-a)} \\
- \frac{\alpha_3}{2\sqrt{a_1}} e^{-\sqrt{a_1} (x_0+a)} \left[ \frac{\lambda_1 \sqrt{a_1} - s \lambda_2 \sqrt{a_2} \sqrt{a_1} - \lambda_2 \sqrt{a_2}}{\lambda_1 \sqrt{a_1} - s \lambda_2 \sqrt{a_2} \sqrt{a_1} + \lambda_2 \sqrt{a_2}} \right] \\
\end{align*}
\]

(70)

yielding for \(x_0>a\),

\[
\tilde{p}_{D_1} = - \frac{\alpha_3}{2\sqrt{a_1}} \left[ e^{-\sqrt{a_1} (x_0-a)} + \left( \frac{\lambda_1 \sqrt{a_1} - s \lambda_2 \sqrt{a_2} \sqrt{a_1} - \lambda_2 \sqrt{a_2}}{\lambda_1 \sqrt{a_1} - s \lambda_2 \sqrt{a_2} \sqrt{a_1} + \lambda_2 \sqrt{a_2}} \right) e^{-\sqrt{a_1} (x_0+a)} \right] \\
\]

(71)
and for \(0 < x_0 < a\),

\[
\bar{P}_{D_1} = D \left[ 1 - \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} \right] \left[ \frac{e^{\sqrt{\alpha_1} x_0} + e^{-\sqrt{\alpha_1} x_0}}{2} \right]
+ D \frac{\lambda_2}{\lambda_1} \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} \left[ \frac{e^{\sqrt{\alpha_1} x_0} - e^{-\sqrt{\alpha_1} x_0}}{2} \right]
\]

(72)

substituting \(D\), given by equation 67, into equation 72,

\[
\bar{P}_{D_1} = \frac{-\lambda_1 \alpha_3 e^{-\sqrt{\alpha_1}}}{\lambda_1 \sqrt{\alpha_1} - s \lambda_2 \sqrt{\alpha_2} \sqrt{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} \left[ 1 - \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} \right] \left[ \frac{e^{\sqrt{\alpha_1} x_0} + e^{-\sqrt{\alpha_1} x_0}}{2} \right]
+ \frac{\lambda_2}{\lambda_1} \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}} \left[ \frac{e^{\sqrt{\alpha_1} x_0} - e^{-\sqrt{\alpha_1} x_0}}{2} \right]
\]

(73)

rearranging,

\[
\bar{P}_{D_1} = -\frac{\lambda_1 \alpha_3}{2(\lambda_1 \sqrt{\alpha_1} - s \lambda_2 \sqrt{\alpha_2} \sqrt{\alpha_1} + \lambda_2 \sqrt{\alpha_2})}
\]

\[
\times \left[ e^{\sqrt{\alpha_1} (x_0 - \alpha)} \left[ 1 - \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} + \frac{\lambda_2 \sqrt{\alpha_2}}{\lambda_1 \sqrt{\alpha_1}} \right] \right.
\]

\[
+ e^{-\sqrt{\alpha_1} (x_0 + \alpha)} \left[ 1 - \frac{\lambda_2}{\lambda_1} \sqrt{\alpha_2} - \frac{\lambda_2 \sqrt{\alpha_2}}{\lambda_1 \sqrt{\alpha_1}} \right]
\]

(74)

simplifying,

\[
\bar{P}_{D_1} = -\frac{\alpha_3}{2 \sqrt{\alpha_1}} e^{\sqrt{\alpha_1} (x_0 - \alpha)} - \frac{\alpha_3}{2 \sqrt{\alpha_1}} e^{-\sqrt{\alpha_1} (x_0 + \alpha)}
\]
\[ \begin{align*}
&= \frac{\lambda_1 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} - \lambda_2 \sqrt{\alpha_2}}{\lambda_2 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} \\
\end{align*} \] (75)

giving the solution to \( P_{D_1} \) at \( 0 < x_D < a \) as,

\[ P_{D_1} = - \frac{\alpha_3}{2 \sqrt{\alpha_1}} \left[ e^{\sqrt{\alpha_1} (x_D - a)} \right. \]

\[ + \left. \frac{\lambda_1 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} - \lambda_2 \sqrt{\alpha_2}}{\lambda_2 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} e^{\sqrt{\alpha_1} (x_D + a)} \right] \] (76)

Combining equation 71 and 76 by the reciprocity principle for \( 0 < x_D < \infty \) gives,

\[ P_{D_1} = - \frac{\alpha_3}{2 \sqrt{\alpha_1}} \left[ e^{-\sqrt{\alpha_1} |x_D - a|} \right. \]

\[ + \left. \frac{\lambda_1 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} - \lambda_2 \sqrt{\alpha_2}}{\lambda_2 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} e^{-\sqrt{\alpha_1} (x_D + a)} \right] \] (77)

and the solution for \( P_{D_2} \), from equation 43, where \( x_D < 0 \),

\[ P_{D_2} = \frac{-\lambda_1 \alpha_3 e^{-\sqrt{\alpha_1}}}{\lambda_1 \sqrt{\alpha_1} - \frac{\lambda_2 \lambda_2 \sqrt{\alpha_1}}{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} e^{\sqrt{\alpha_1} x_D} \] (78)
V. DISCUSSION

Evaluating the analytical solutions, equation 77 and 78, requires inversion through Fourier space and Laplace space back into real space. To accomplish this inversion two programs were developed, the Fourfirst Program which inverts in Fourier space first and the Stehfirst Program which inverts in Laplace space first. (A full discussion of the mechanics of each program is presented in chapter VI.) Comparison of the two programs showed no significant computing advantage of one over the other; if anything, the Stehfirst Program executed approximately 15% faster, in cpu time.

Since the pressure function derived in the analytical solution tends toward oscillation, due to the fourier cosine summation, a convergence criteria was needed to keep check on the values produced from the fourier inversion. The convergence was set to $10^{-14}$, meaning that relative changes between summation values needed to change less than $10^{-14}$ before the value was passed to or from the Laplace inverter. Figure 2 shows the behavior of the initial convergence where $t_0 = 0.1$.

After evaluating the soundness of the programs, delimiting parameters were placed on the solution. (See Appendix A for the analytical delimiting solution showing line source concurrence.) The pressure point was situated one wellbore radius away from the well and the vertical no-flow boundary was placed some distance away from the well; equidistant above and below the well. The boundary skin was set to zero, mobility ratios between the two regions were set equally at one as were the diffusivity ratios. Inputting this criteria into the numerical inversions produced the plot shown in figure 3. Figure 3 shows the vertical no-flow boundary, $y^b_D$, at 20 with the well located at $a = 6$, $b = 10$; the pressure point at $x_D = 7$, $y_D = 10$. No variation between programs was exhibited and as expected, early time data followed strictly along the line source solution.
Figure 2: Semilog plot depicting the behavior of the pressure function during numerical inversion.
Figure 3: Logarithmic plot of $p_D$ vs. $t_D$ for values generated at $y^*_D = 20$ in the Fourfirst and Stehfirst Programs.
Figure 4 is the standard line source solution for varying values of $y_0^D$. This array of curves was generated for $10 < y_0^D < 10000$ using the equation

$$p_D = \frac{1}{2} \sum_{i=1}^{n} Ei \left( -\frac{y_0^D D_i}{4l_D} \right) \frac{1}{4} Ei \left( -\frac{1}{4l_D} \right).$$

Using figure 4 for comparison, data was generated using the numerically inverted solution for similar values of $y_0^D$. Figure 5 shows these results. As is apparent, no match can be made except for $y_0^D = 20$. Using a shifting method the curves were found to follow those in figure 4. Figure 6 shows the shifted results. The shift used was a factor of $y_0^D / 20$. The fact that this shift was necessary shows an as yet unresolved puzzle in the analytical solution. This is under further consideration.

Since the data for $y_0^D = 20$ shows matching early time data to the line source and half slope behavior at long time, it was used with the line source as a base for the next two cases -- simulation of infinite mobility and infinite skin. To show the behavior of an impermeable boundary two alternatives were explored. Figure 7 concerns a high boundary skin, skin=10000. This prohibits communication between region I and II. The late time pressure response doubled that of the open, flowing system. This concurs with the response typically seen with no-flow boundaries such as vertical fractures or sealing faults. This response can similarly be used to evaluate the length to the barrier based on the length of the half slope period. The second case investigated was setting the mobility of fluids in region I infinitely high, $\lambda_1 = 1000$, such that flow with the low mobility fluids of region II was virtually stopped, thus forming a no-flow barrier. Figure 8 shows the anticipated late time half slope curve with double the pressure values of the late time curve with no fluid barrier. (Well location, pressure point location and diffusivities were not changed in either case.)

Appendix C contains selected data for each case discussed herein.
Figure 4: Logarithmic plot of the sum of line sources with a well located in the center of the system.

\[ p_D = \frac{1}{2} \sum_{i=1}^{n} Ei(-\frac{y^2 D^2}{4l_D}) - \frac{1}{2} Ei(-\frac{1}{4l_D}) \] where \( y^* D \) ranges from 10 to 10000.
Figure 5: Logarithmic plot of $p_D$ vs. $t_D$ for composite reservoir with $y^*D$ varying from 10 to 200.
Figure 6: Logarithmic plot of $p_D$ vs. $t_D$ shifted data from the composite system.
Figure 7: Logarithmic plot for case of infinite and zero boundary skin.
Figure 3: Logarithmic plot for case of infinite and unit mobility.
VI. COMPUTATIONAL PROCEDURES

Exact values for the integrals in equations 77 and 78 are evaluated numerically using either of the programs listed in Appendix B. The two programs differ in order of inversion of the problem. The Stehfist Program inverts the problem in time prior to Fourier inversion of the y variable. The Fourfirst Program, on the other hand, inverts the y variable prior to the Laplace variable.

Numerical inversion of the governing Laplace equations is facilitated by using an algorithm developed by Stehfest [1970]. A Fourier inversion algorithm was written to complement the Stehfest algorithm. The Fourier inverse transform as defined by Selby [1970] is the basis for the algorithm.

The programs are intentionally general such that input data can tailor the problem to be solved. For example, the mobility ratio can be fixed while varying skin in order to observe the pressure effects over time. Additionally, values for pressure are generated on a logarithmic time scale to ease graphing and lessen the accumulation of data once a trend has been established.

Flowcharts for each program are presented in Appendix B preceding the coding.
VII. CONCLUSIONS AND RECOMMENDATIONS

The objective of this research was to develop a mathematical solution for evaluating transient pressure in a strip reservoir which contains a linear skin discontinuity. An analytical solution was presented with two integral transformations which, due to complexity, were inverted numerically. In evaluating the order of numerical inversion, it was found that using the time inversion prior to space inversion held no advantages over space inversion prior to time inversion.

Verification of results was made by matching the solution to a hypothetical infinite system with the well located centrally and pressure response measured one radii away from the well. It was found for cases of varying vertical no-flow boundaries, the $\frac{\gamma_D}{20}$ case matched; thus leading to a constant shift in data of $\frac{\gamma_D}{20}$ for the accompanying curves. A closer look at the cause of this shift is under consideration. An impermeable boundary was set in two cases by equating mobility ratios to 1000 and, in the second case, by setting the boundary skin to 10000. Late time pressure response concurred doubling of the half slope period as expected.

Future studies using this work as a template can be made studying the effects of varying parameters on the pressure response of the composite infinite strip with a skin boundary.
NOMENCLATURE

\( a \) = \( x \) coordinate for well location
\( b \) = \( y \) coordinate for well location
\( C \) = constant in the general solution
\( c \) = compressibility
\( D \) = constant in the general solution
\( F \) = Fourier transform operator
\( h \) = formation thickness
\( k \) = permeability
\( L \) = Laplace transform operator
\( m \) = Fourier variable
\( p \) = pressure
\( \bar{p} \) = Laplace transform of \( p \)
\( \mathcal{F} \) = Fourier transform of \( \bar{p} \)
\( \mathcal{F} \) = Laplace transform of \( \bar{p} \)
\( \Delta p_{\text{skin}} \) = pressure drop across the skin boundary
\( \dot{a} \) = volumetric rate
\( q(t) \) = time varying production rate
\( r \) = radius
\( s \) = dimensionless skin
\( S \) = Laplace variable for the time domain
\( t \) = time
\( u \) = Laplace variable for the \( x \) domain
\( x \) = \( x \) coordinate of the pressure point
\( y \) = \( y \) coordinate of the pressure point
\( y_0 \) = finite width of the reservoir
\( Y \) = variable transform for \( y \), \( \pi y_0 D \)

\[
\begin{align*}
\alpha_1 & = \sqrt{m^2 + S} \\
\alpha_2 & = \sqrt{m^2 + \frac{\eta_1}{\eta_2} S} \\
\alpha_3 & = \frac{\cos(vmb)}{S} \\
\delta & = \text{Dirac delta function} \\
\eta & = \text{diffusivity constant} \\
\lambda & = \text{mobility} \\
\mu & = \text{fluid viscosity} \\
\nu & = \pi y_0 D \\
\phi & = \text{formation porosity (fraction)} \\
\pi & = \pi, \ 3.14152976
\end{align*}
\]

Subscript

\( 1 \) = region 1, \( x>0 \)
\( 2 \) = region 2, \( x<0 \)
\( D \) = dimensionless
\( i \) = initial
\( t \) = total system
\( w \) = wellbore
REFERENCES


APPENDIX A: Delimiting Solution For Line Source Concurrence
Delimiting Solution for Line Source Concurrence

In evaluating the validity of the derived solutions, a reduction of the Solution can be made such that a comparison with the line source solution is possible. Assuming the conditions the line source was developed under will be repeated for equation 71 where \( x_D > a \).

Isolating equation 71,

\[
\bar{D}_{D1} = \frac{-\alpha_3}{2\sqrt{\alpha_1}} \left[ e^{-\sqrt{\alpha_1}(x_D - a)} + \frac{\lambda_1 \sqrt{\alpha_1} - s \lambda_2 \sqrt{\alpha_2} \sqrt{\alpha_1} - \lambda_2 \sqrt{\alpha_2}}{\lambda_1 \sqrt{\alpha_1} - s \lambda_2 \sqrt{\alpha_2} \sqrt{\alpha_1} + \lambda_2 \sqrt{\alpha_2}} \right] e^{-\sqrt{\alpha_1}(x_D + a)}
\]

letting \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) revert to their substituting parameters and assuming the following:

\[
\begin{align*}
\lambda_1 \lambda_2 &= \eta_1 \eta_2 = 1 \\
Y_0 &= W^2
\end{align*}
\]

equation 71 reduces to

\[
\bar{D}_{D1} = \frac{-e^{-\sqrt{\lambda_1}m^2 + S} \cos(vmb)}{2\sqrt{\lambda_1}m^2 + S}
\]

Recalling the Finite Fourier Cosine Inverse Transformation from Selby [1970] defined for \( 0 < Y_D < \pi \),

\[
\bar{F}(Y) = \frac{1}{\pi} f_c(0) - \frac{2}{\pi} \sum_{m=1}^{\infty} f_c(m) \cos(mY_D)
\]

where \( Y_D \) will be defined in the center of the system at \( Y_D = \pi/2 \).

Applying equation 83 to equation 82 gives,

\[
\bar{D}_{D1} = -\frac{e^{-\sqrt{\lambda_1}m^2 + S}}{2\pi \sqrt{\lambda_1}m^2 + S} - \frac{2}{\pi} \sum_{m=1}^{\text{even}} \frac{e^{-\sqrt{\lambda_1}m^2 + S} \cos(vmb) \cos(mY_D)}{2}
\]
letting

\[ w = vm \]  \hspace{1cm} (85)

equation 84 becomes:

\[
\bar{F}_{D_t} = - \frac{e^{-\sqrt{S}}}{2\pi \sqrt{S}} - \frac{2}{\pi} \sum_{\text{mean}} \frac{e^{-\sqrt{w+S}}}{\sqrt{w+S}} \frac{\cos(wb) \cos(mY_D)}{2}
\]  \hspace{1cm} (86)

Recalling the following Laplace inverse transformations from Churchill [1944]:

\[
L^{-1} \left[ e^{-\sqrt{S}} \right] = 2\sqrt{\frac{1}{\pi}} e^{-\frac{1}{4t}} - \text{erfc} \left( \frac{1}{2\sqrt{t}} \right)
\]  \hspace{1cm} (87)

\[
L^{-1} \left[ \frac{e^{-\sqrt{S}}}{\sqrt{S}} \right] = \frac{1}{\sqrt{\pi t}} e^{-\frac{w_1^2}{4t}} = F(t)
\]  \hspace{1cm} (88)

and

\[
L^{-1} \left[ -\frac{\cos(wb) \cos(mY_D)}{2S} \right] = -\frac{\cos(wb) \cos(mY_D)}{2}
\]  \hspace{1cm} (89)

letting \( w_1 = 1 \) and applying the following substitution,

\[
f(S+w^2) = \int_0^\infty e^{-w_1^2 S} F(t) e^{-St} dt
\]  \hspace{1cm} (90)

such that equation (88) takes the form,

\[
L^{-1} \left[ \frac{e^{-\sqrt{w_1^2 S+w^2}}}{\sqrt{w_1^2 S+w^2}} \right] = e^{-\sqrt{w_1}} \left( \frac{1}{\sqrt{\pi t}} \right)
\]  \hspace{1cm} (91)
then, in real space and time, equation 86 becomes:

\[
p_{D_1} = \frac{1}{2\pi} \left[ 2\sqrt{\frac{L_D}{\pi}} e^{-\frac{1}{4L_D}} - \text{erfc}\left(\frac{1}{2\sqrt{L_D}}\right) \right] 
- \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos(wb) \cos(mL_D)}{2\sqrt{\pi}} \int_0^{\infty} e^{-\omega^2 + \frac{1}{4L_D\tau}} d\tau
\]  

(92)

Evaluation of the integral in equation 92 is facilitated by use of the following substitutions from *Abramowitz and Stegun* [1970]:

let

\[
\omega = w
\]

(93)

\[
\beta = \frac{1}{2}
\]

(94)

\[
\chi^2 = \tau
\]

(95)

and

\[
d\tau = 2\chi d\chi
\]

(96)

Applying these substitutions to the following integral defined in *Abramowitz and Stegun* [1970] yields,

\[
\int e^{-\omega^2\chi^2 - \frac{\beta^2}{\chi^2}} d\chi = \frac{\sqrt{\pi}}{4\omega} \left[ e^{2\omega \beta} \text{erf}(\omega \chi + \beta) + \text{const} \right]
\]

(97)

Equation 97 can be directly applied to the integral in equation 92 and evaluated at the limits of 0 to \(t_D\):
Reverting the substitution of \( w \) for \( \omega \) and \( \frac{1}{2} \) for \( \beta \) gives,

\[
\int_{0}^{\infty} e^{-\frac{\omega^2 \tau}{2} - \frac{1}{4t_D}} d\tau = \frac{\sqrt{\pi}}{2w} \left[ e^{\omega} \text{erf}(\omega t_D + \frac{1}{2t_D}) + e^{-\omega} \text{erf}(\omega t_D - \frac{1}{2t_D}) \right]
\]  

(99)

Rearranging equation 99 and substituting \( v_m \) back for \( w \) gives the following solution:

\[
P_{D_1} = -\frac{1}{2\pi} \left[ 2\sqrt{\frac{t_D}{\pi}} e^{-\frac{1}{2t_D}} - \text{erfc}(\frac{1}{2\sqrt{t_D}}) \right] \frac{1}{2\pi} \]

\[
* \sum_{m=\text{even}} \frac{1}{v_m} \left[ e^{v_m} \text{erf}(v_m t_D + \frac{1}{2t_D}) + e^{-v_m} \text{erf}(v_m t_D - \frac{1}{2t_D}) \right]
\]

(100)
APPENDIX B: Computer Programs
General Input:
define regions I, II, and system geometry

Ln(td) increments

Stehfest Algorithm:
for Laplace Inversion
pwd(td,n,alpha)
n=8 parameters

Laplace Function:
sum1(arg)

Fourier Inversion:
\(f(\alpha, \arg)\)
\(xd > 0, xd < 0\)

Figure 9: Flowchart for Fourfirst Program
This program generates $pd$ versus $td$ for a system containing a linear skin discontinuity. Numerical inversion of $pd$ is accomplished by first inverting the Fourier function in the $y$ domain then inverting the Laplace function in time.

Variable Definitions:

- $n$ = number of parameters for the Stehfest Algorithm
- $m$ = calculation flag
- $td$ = dimensionless time
- $pdw$ = dimensionless pressure
- $pwd$ = function called within Stehfest that contains Laplace equation to be inverted
- $pwdl$ = Stehfest function that inverts Laplace equation
- $sum1$ = function called within Fourier Inverter containing the Finite Fourier Cosine equation to be inverted
- $f$ = Fourier function that inverts the Finite Fourier Cosine equation
- $arg$ = Laplace variable
- $alpha$ = Fourier variable
- $xd$ = dimensionless $x$ coordinate defining the pressure point
- $yd$ = dimensionless $y$ coordinate defining the pressure point
- $a$ = dimensionless $x$ coordinate defining the well location
- $b$ = dimensionless $y$ coordinate defining the well location

Input required by the program is read in from a file called 'damin'. Contents of the data file are the well and pressure locations, skin, dimension of the system in the $y$ direction, and formation properties.

Output appears in a file called 'data.out' which contains the values for $pd$ vs. $td$.

```fortran
implicit real*8(a-h,o-z)
real*8 nu,i,k,l,mm,nn
common a,b,yodd,skin,xd,yd,eta1,eta2,alam1,alam2,m
open(unit=3, file='data.out')
rewind(unit=3)
open(unit=4, file='damin', status='old', access='sequential')
rewind(unit=4)
read(4,*),a,yodd,skin,xd,yd,eta1,eta2,alam1,alam2
write(3,7),a,yodd,skin,xd,yd,eta1,eta2,alam1,alam2
7 format(5x,'data: a=',f3.1,'x',b=','f4.1,1x,'yod=','f6.1,1x,'+skin=','f3.1,1x,'xd=','f3.1,1x,'yd=','f4.1,1x,12x,'eta1=','+f3.1,1x','eta2=','f3.1,1x','alam1=','f3.1,1x','alam2=','f3.1')
```
pi = 3.14152976

nu = pi/yod

m = 777
n = 8

do 10 i = 0, 6
    k = 10**i
    l = k*10
    mm = 2*k

do 20 nn = mm, l, k
    td = nn
    call pwd(td, n, pdw)
    write(3, *) td, pdw
20   continue
10   continue

stop
end

C

THE STEHFEST ALGORITHM

SUBROUTINE PWD(TD, N, PD)

THIS FUNCTION COMPUTES NUMERICALLY THE LAPLACE TRANSFORM INVERSE OF F(S).

IMPLICIT REAL*8 (A-H, O-Z)
DIMENSION G(50), V(50), H(25)
common a, b, yod, skin, xd, yd, eta, eta2, alam, alam2, m

NOW IF THE ARRAY V(I) WAS COMPUTED BEFORE THE PROGRAM GOES DIRECTLY TO THE END OF THE SUBROUTINE TO CALCULATE F(S).
IF (N .EQ. M) GO TO 17
M = N
DLOGTW = 0.6931471805599
NH = N/2

THE FACTORIALS OF 1 TO N ARE CALCULATED INTO ARRAY G.
G(1) = 1
DO 1 I = 2, N
    G(I) = G(I-1)*I
1   CONTINUE

TERMS WITH K ONLY ARE CALCULATED INTO ARRAY H.
H(1) = 2/G(NH-1)
DO 6 I = 2, NH
6   continue
FI-I
IF(I-NH) 4,5,6
4
H(I)=FI**NH*G(2*I)/(G(NH-I)*G(I)*G(I-1))
GO TO 6
5
H(I)=FI**NH*G(2*I)/(G(I)*G(I-1))
6 CONTINUE
C C THE TERMS (-1)**NH+1 ARE CALCULATED.
C C FIRST THE TERM FOR I=1
SN=2*(NH-NH/2*2)-1
C C THE REST OF THE $S_N$'s ARE CALCULATED IN THE MAIN RUTINE.
C C THE ARRAY V(I) IS CALCULATED.
DO 7 1,1,N
C C FIRST SET V(I)=0
V(I)=0.
C C THE LIMITS FOR K ARE ESTABLISHED.
C C THE LOWER LIMIT IS K1=INT((I+1/2))
K1=(I+1)/2
C C THE UPPER LIMIT IS K2=MIN(I,N/2)
K2=I
IF (K2-NH) 8,8,9
9
K2=NH
C C THE SUMMATION TERM IN V(I) IS CALCULATED.
DO 10 K=K1,K2
10 IF (2*K-I) 12,13,12
11 V(I)=V(I)+H(K)/(G(I-K)*G(2*K-I))
GOTO 10
13 V(I)=V(I)+H(K)/G(I-K)
GOTO 10
14 V(I)=V(I)+H(K)/G(2*K-I)
CONTINUE
C C THE V(I) ARRAY IS FINALLY CALCULATED BY WEIGHTING
C C ACCORDING TO SN.
V(I)=SN*V(I)
C C THE TERM SN CHANGES ITS SIGN EACH ITERATION.
SN=-SN
CONTINUE
C C THE NUMERICAL APPROXIMATION IS CALCULATED.
PD=0.0
PMCA=DLOGTW/TD
DO 15 I=1,N
ARG=PMCA*I
funs=suml(arg)
function sum (arg)
implicit real*8(a-h,o-z)
real*8 nu
common a,b,yod,skin,xd,yd,eta1,eta2,alam1,alam2,m

pi=3.1415926
nu=pi/yod
yyd=pi*yd/yod
argo=0.0
fun1=f(arg0,arg)
sumo=fun1
sum=sumo
sum2=10
sum3=10

starting the computation loop
do 20 j=1,5000
arg2=j
fun2=f(arg2,arg)
sum5=fun2*cos(j*yyd)
sum4=sum2
sum2=sum3
sum3=sum5
sum=sum+sum5
adelta=abs(sum3)+abs(sum2)+abs(sum4))/sum
delta=abs(adelta)
if(delta.lt.0.00000000000001) goto 999
20 continue

999 sum=-(1.0/pi)*fun1+(2.0/pi)*(sum-fun1))

return
end

double precision function f(alpha,arg)
implicit real*8 (a-h,o-z)
real*8 nu
common a,b,yod,skin,xd,yd,eta1,eta2,alam1,alam2,m
\[ \pi = 3.1415926 \]
\[ \nu = \pi / \text{yod} \]

\[ f_1 = \sqrt{\nu \nu \alpha \alpha + \text{arg}} \]
\[ f_2 = \sqrt{\nu \nu \alpha \alpha + \eta_1 / \eta_2 \text{arg}} \]
\[ f_3 = \alpha_1 f_1 - \text{skin} \alpha_2 f_2 + \alpha_1 f_2 \]
\[ f_4 = \alpha_1 f_1 - \text{skin} \alpha_2 f_2 + \alpha_1 f_2 \]

if (\( x_d > 0.0 \)) then
\[ f = (-\cos(\nu \alpha b / (2.0 \text{arg} f_1)) * (dexp(-f_1 \text{abs}(x_d - a) + f_4 / f_3) * dexp(-f_1 (x_d + a)))) \]
else
\[ f = (-\alpha_1 \cos(\nu \alpha b) * dexp(f_2 \text{abs} - f_1 \text{f})) / (\text{arg} f_3) \]
endif

return
end
General Input: define regions I, II, and system geometry

Time Loop: ln(td) increments

Fourier Inversion: sum1(td,n)

Stehfest Algorithm: for Laplace Inversion
pwd(td,n,alpha)
n=8 parameters

Laplace Function: xd>0, xd<0
f(alpha, arg)

output: td, pd

Figure 10: Flowchart for Stehfirst Program
PROGRAM STEHFIRST

This program generates \( pd \) versus \( td \) for a system containing a linear skin discontinuity. Numerical inversion of \( pd \) is accomplished by first inverting the Laplace function in time then inverting the Finite Fourier function in the \( y \) domain.

Variable Definitions:

- \( n \) = number of parameters for the Stehfest Algorithm
- \( m \) = calculation flag
- \( td \) = dimensionless time
- \( pd \) = dimensionless pressure
- \( pwd \) = function called within Stehfest that contains Laplace equation to be inverted
- \( pwdl \) = Stehfest function that inverts Laplace equation
- \( sum1 \) = function called within Fourier Inverter containing the Finite Fourier Cosine equation to be inverted
- \( f \) = Fourier function that inverts the Finite Fourier Cosine equation
- \( arg \) = Laplace variable
- \( alpha \) = Fourier variable
- \( xd \) = dimensionless \( x \) coordinate defining the pressure point
- \( yd \) = dimensionless \( y \) coordinate defining the pressure point
- \( a \) = dimensionless \( x \) coordinate defining the well location
- \( b \) = dimensionless \( y \) coordinate defining the well location

Input required by the program is read in from a file called 'data.in'. Contents of the data file are the well and pressure locations, skin, dimension of the system in the \( y \) direction, and formation properties.

Output appears in a file called 'data.out' which contains the values for \( pd \) vs. \( td \).

implicit real*8(a-h,o-z)
real*8 nu,i,k,l,mm,nn
common a,b,yod,skin,xd,yd,eta 1,eta 2,alam1,alam2,m

open(unit=3, file='data.out')
rewind(unit=3)
open(unit=4, file='data.in', status='old', access='sequential')
rewind(unit=4)

read(4,*) a,b,yod,skin,xd,yd,eta 1,eta 2,alam1,alam2
write(3,7)a,b,yod,skin,xd,yd,eta 1,eta 2,alam1,alam2

7 format(5x,'data: a=',f3.1lx,'b=',f4.1lx,'yod=',f6.1lx,
+ 'skin=',f3.1lx,'xd=',f3.1lx,'yd=',f4.1lx,12x,'eta1=',
+ f3.1lx,'eta2=',f3.1lx,'alam1=',f3.1lx,'alam2=',f3.1)'
pi = 3.14152976
nu = pi/yod

m = 777
n = 8
do 10 i = 0, 6
  k = 10**i
  l = k * 10
  mm = 2 * k

  do 20 nn = mm, l, k
    td = nn

    pd = sum1(td, n)
    write(3, *) td, pd
    print *, td, pd

20    continue
10    continue

stop
end

function sum1(td, n)
implicit real*8(a-h,o-z)
real*8 nu
common a, b, yod, skin, xd, yd, eta1, eta2, alam1, alam2, m

pi = 3.14152976
nu = pi/yod
yyd = pi*yd/yod
arg0 = 0.0
fun1 = pwd(td, n, arg0)
sum = fun1
sum2 = 10
sum3 = 10


starting the computation loop

  do 20 j = 1, 5000
    arg2 = j
    fun2 = pwd(td, n, arg2)
    sum5 = fun2*cos(j*yyd)
    sum4 = sum2
    sum2 = sum3
    sum3 = sum5
    sum = sum + sum5

    adelma = (abs(sum3) + abs(sum2) + abs(sum2))/sum
    delta = abs(adelma)
    if(delta > 0.00000000000001) goto 999
THE STEHFEST ALGORITHM

FUNCTION PWD(TD,N,ALPHA)
    THIS FUNCTION COMPUTES NUMERICALLY THE LAPLACE TRANSFORM INVERSE OF F(S).
    IMPLICIT REAL*8(A-H,O-Z)
    DIMENSION G(50),V(50),H(25)
    common a,b,yod,skin,xd,yd,eta1,eta2,alam1,alam2,m

    NOW IF THE ARRAY V(I) WAS COMPUTED BEFORE THE PROGRAM GOES DIRECTLY TO THE END OF THE SUBROUTINE TO CALCULATE F(S).
    IF (N.EQ.M) GO TO 17
    M-N
    DLOGTW=0.6931471805599
    NH=N/2

    THE FACTORIALS OF 1 TO N ARE CALCULATED INTO ARRAY G.
    G(1)=1
    DO 1 I=2,N
        G(I)=G(I-1)*I
    1    CONTINUE

    TERMS WITH K ONLY ARE CALCULATED INTO ARRAY H.
    H(1)=2./G(NH-1)
    DO 6 I=2,NH
        H(I)=F(I)/G(I)*G(I-I)
    4    IF(I-NH) 5,6
    5    H(I)=F*I**H*G*(2*I)/(G(I)*G(I-I))
    6    CONTINUE

    THE TERMS (-1)**NH+1 ARE CALCULATED.
    FIRST THE TERM FOR I=1
    SN=2*(NH-NH/2*2)-1
THE REST OF THE SN'S ARE CALCULATED IN THE MAIN RUTINE.

THE ARRAY V(I) IS CALCULATED.

DO 7 I=1,N

FIRST SET V(I)=0
V(I)=0.

THE LIMITS FOR K ARE ESTABLISHED.

THE LOWER LIMIT IS K1=INTEG((I+1)/2)
K1=(I+1)/2

THE UPPER LIMIT IS K2=MIN(I,N/2)
K2=1
IF (K2-NH) 8,8,9
K2=NH

THE SUMMATION TERM IN V(I) IS CALCULATED.

DO 10 K=K1,K2

IF (2*K-I) 12,13,12
IF (I-K) 11,14,11
V(I)=V(I)+H(K)/(G(I-K)*G(2*K-I))
GOTO 10
V(I)=V(I)+H(K)/G(I-K)
GOTO 10
V(I)=V(I)+H(K)/G(2*K-I)
CONTINUE

THE V(I) ARRAY IS FINALLY CALCULATED BY WEIGHTING
ACCORDING TO SN.

V(I)=SN*V(I)

THE TERM SN CHANGES ITS SIGN EACH ITERATION.

SN=-SN
CONTINUE

THE NUMERICAL APPROXIMATION IS CALCULATED.

PWD=0.0
PMCA=DLOGTW/TD
DO 15 I=1,N
ARG=PMCA*I
funs=f(alpha,ARG)
PWD=PWD+V(I)*funs
CONTINUE
PWD=PWD*PMCA
RETURN
END
function f(alpha, arg)
implicit real*8 (a-h, o-z)
real*8 nu
common a, b, yod, skin, xd, yd, eta1, eta2, alam1, alam2, m
pi=3.14159265
nu=pi/yod

f1=sqrt(nu*nu*alpha*alpha+arg)
f2=sqrt(nu*nu*alpha*alpha+eta1/eta2*arg)
f3=alam1*f1-skin*alam2*f2*f1+alam2*f2
f4=alam1*f1-skin*alam2*f2*f1-alam2*f2

if(xd.gt.0.0) then
  f=(-cos(nu*alpha*b)/(2.0*arg*f1))*dexp(-f1*abs(xd-a))
  f*=f4/f3*dexp(-f1*(xd+a))
else
  f=-alam1*cos(nu*alpha*b)*dexp(f2*xd-a*f1)/(arg*f3)
endif

return
end
TABLE C-1

Selected Data from Stehfirst and Fourfirst Program Evaluation

<table>
<thead>
<tr>
<th></th>
<th>$t_D$</th>
<th>$P_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stehfirst Program</strong>:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y^0_D = 20$</td>
<td>0.10</td>
<td>0.0125</td>
</tr>
<tr>
<td>skin = 0</td>
<td>0.30</td>
<td>0.1485</td>
</tr>
<tr>
<td>$a = 6$</td>
<td>1.00</td>
<td>b.5289</td>
</tr>
<tr>
<td>$b = 10$</td>
<td>3.00</td>
<td>1.0077</td>
</tr>
<tr>
<td>$x_D = 7$</td>
<td>10.00</td>
<td>1.5891</td>
</tr>
<tr>
<td>$y_D = 10$</td>
<td>30.00</td>
<td>2.1454</td>
</tr>
<tr>
<td>$\lambda_1 = \lambda_2 = \eta_1 = \eta_2 = 1$</td>
<td>100.00</td>
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</tr>
<tr>
<td></td>
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<tr>
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<td>1000.00</td>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>10000.0</td>
<td>19.1288</td>
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<td>36.2759</td>
</tr>
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<td>100000.0</td>
<td>57.9617</td>
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<tr>
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<td>W.5366</td>
</tr>
<tr>
<td></td>
<td>900000.0</td>
<td>171.546</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$t_D$</th>
<th>$P_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fourfirst Program</strong>:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y^0_D = 20$</td>
<td>0.10</td>
<td>0.0125</td>
</tr>
<tr>
<td>skin = 0</td>
<td>0.30</td>
<td>0.1485</td>
</tr>
<tr>
<td>$a = 6$</td>
<td>1.00</td>
<td>10.5289</td>
</tr>
<tr>
<td>$b = 10$</td>
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</tr>
<tr>
<td>$x_D = 7$</td>
<td>10.00</td>
<td>1.5891</td>
</tr>
<tr>
<td>$y_D = 10$</td>
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<td>2.1454</td>
</tr>
<tr>
<td>$\lambda_1 = \lambda_2 = \eta_1 = \eta_2 = 1$</td>
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<td>12.9698</td>
</tr>
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<tr>
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</table>
TABLE C-2

Selected Data from Composite Reservoir -- Nonshifted

\[ \gamma_D^* = 20 \]
\[ \text{skin} = 0 \]
\[ a = 6 \]
\[ b = 10 \]
\[ x_D = 7 \]
\[ y_D = 10 \]
\[ \lambda_1 = \lambda_2 = \eta_1 = \eta_2 = 1 \]

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<th>( t_D )</th>
<th>( P_D )</th>
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<tr>
<td>0.10</td>
<td>0.0125</td>
</tr>
<tr>
<td>0.30</td>
<td>0.1485</td>
</tr>
<tr>
<td>1.00</td>
<td>0.5289</td>
</tr>
<tr>
<td>3.00</td>
<td>1.0777</td>
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<td>10.00</td>
<td>1.5891</td>
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</table>

\[ \gamma_D^* = 200 \]
\[ \text{skin} = 0 \]
\[ a = 6 \]
\[ b = 10 \]
\[ x_D = 7 \]
\[ y_D = 10 \]
\[ \lambda_1 = \lambda_2 = \eta_1 = \eta_2 = 1 \]

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### TABLE C-3

Selected Data from Composite Reservoir $\cdot$ Shifted

\[
y^D = 20
\]

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<th>$x_D$</th>
<th>$y_D$</th>
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<tbody>
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<td>7</td>
<td>10</td>
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</table>

<table>
<thead>
<tr>
<th>$t_D$</th>
<th>$P_D$</th>
</tr>
</thead>
<tbody>
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<td>0.10</td>
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<tr>
<td>0.30</td>
<td>0.1485</td>
</tr>
<tr>
<td>1.00</td>
<td>0.5289</td>
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<td>1.0077</td>
</tr>
<tr>
<td>10.00</td>
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\[
y^D = 200
\]

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TABLE C-4

Selected Data for Varying Mobility Ratios and Skin

Case 1: High Mobility Ratio

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| $\lambda_2$ = 1 | | $\eta_1 = 12 = 1$ |
|-----------------|-----------------|
| $\eta_1$ = 12 = 1 |        |

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Case 2: High Boundary Skin

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TABLE C-5

The Line Source Solution

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